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A COROLLARY OF THE B-FUNCTION LEMMA

A. BEILINSON AND D. GAITSGORY

1. THE STATEMENT

1.1. Let X be a smooth algebraic variety over an algebraically closed field k of characteristic 0. Let f be a function on X ; let Y be the locus of zeros of f , and $j : U \hookrightarrow X$ the open embedding of the complement of Y . Let D_X be the sheaf of differential operators on X , and let \mathcal{M} be a holonomic (left) D -module on U .

Let us tensor D_X with the ring of polynomials in one variable $k[s]$. I.e., let us consider the sheaf $D_X[s]$, and the corresponding category of (left) $D_X[s]$ -modules (we follow the conventions in the theory of D -modules, where we only consider sheaves of D_X - or $D_X[s]$ -modules that are quasi-coherent as sheaves of \mathcal{O}_X -modules).

Consider now the $D_U[s]$ -module “ fs ”. By definition, as $\mathcal{O}_U[s]$ module, it is free of rank one with the generator that we denote f^s , and vector fields acting on it by the formula

$$\xi(f^s) = s \cdot \xi(f) \cdot f^{s-1},$$

where $f^{s-1} := f^{-1} \cdot f^s$.

Consider the $D_U[s]$ -module $\mathcal{M} \otimes \text{“}fs\text{”} := \mathcal{M} \otimes_{\mathcal{O}_U} \text{“}fs\text{”}$, and the $D_X[s]$ -module

$$j_*(\mathcal{M} \otimes \text{“}fs\text{”}).$$

It is easy to see that in general $j_*(\mathcal{M} \otimes \text{“}fs\text{”})$ is not finitely generated as a $D_X[s]$ -module:

Example. Consider $X = \mathbb{A}^1 := \text{Spec}(k[t])$, $f = t$, $\mathcal{M} = \mathcal{O}_X$. Let $\tilde{\mathcal{M}}$ be the $D_X[s]$ -submodule of $j_*(\text{“}fs\text{”})$, generated by the section f^s . It is easy to see that we have an isomorphism

$$j_*(\text{“}fs\text{”})/\tilde{\mathcal{M}} \simeq \bigoplus_{n=0,1,2,\dots} (\delta_0 \otimes (k[s]/s - n)),$$

where δ_0 is the δ -function at $0 \in \mathbb{A}^1$, thought of as a left D -module on \mathbb{A}^1 , and $n \in \mathbb{N}$ is regarded as a point of $k \subset \text{Spec}(k[s])$.

1.2. The goal of this note is to describe the set $\mathbf{V}(\mathcal{M})$ of all $D_X[s]$ -submodules $\tilde{\mathcal{M}} \subset j_*(\mathcal{M} \otimes \text{“}fs\text{”})$, such that $j^*(\tilde{\mathcal{M}}) = \mathcal{M} \otimes \text{“}fs\text{”}$, and the subset $\mathbf{V}_f(\mathcal{M}) \subset \mathbf{V}(\mathcal{M})$ that corresponds to those $\tilde{\mathcal{M}}$ that are finitely generated as $D_X[s]$ -modules.

For $\tilde{\mathcal{M}} \in \mathbf{V}(\mathcal{M})$ and a point $\lambda \in k \subset \text{Spec}(k[s])$ consider the D_X -module $\tilde{\mathcal{M}}_\lambda := \tilde{\mathcal{M}}/(s - \lambda)$. We have the canonical maps

$$j_!(\mathcal{M} \otimes \text{“}f^\lambda\text{”}) \rightarrow \tilde{\mathcal{M}}_\lambda \rightarrow j_*(\mathcal{M} \otimes \text{“}f^\lambda\text{”}),$$

where $\mathcal{M} \otimes \text{“}f^\lambda\text{”} := \mathcal{M} \otimes_{\mathcal{O}_U} \text{“}f^\lambda\text{”}$ denotes the corresponding D -module over U .

To state our main result, we shall adopt the following conventions. By an arithmetic progression in k we shall mean a coset of k modulo \mathbb{Z} . Let $\Lambda \subset k$ be a subset equal to union of finitely many arithmetic progressions. We say that some property of an element of Λ holds for $\lambda \gg 0$ (resp., $\lambda \ll 0$), if it holds for elements of the form $\lambda_0 + n$ for any fixed $\lambda_0 \in \Lambda$, whenever $n \in \mathbb{Z}$ is sufficiently large (resp., small).

We now are ready to state our theorem:

Theorem 1. *There exist a subset $\Lambda \subset k$ equal to the union of finitely many arithmetic progressions such that for any $\tilde{\mathcal{M}} \in \mathbf{V}_f(\mathcal{M})$ we have:*

(1) *For $\lambda \notin \Lambda$ the maps*

$$j_!(\mathcal{M} \otimes "f^\lambda") \rightarrow \tilde{\mathcal{M}}_\lambda \rightarrow j_*(\mathcal{M} \otimes "f^\lambda")$$

are isomorphisms. In particular, $\tilde{\mathcal{M}}_\lambda \simeq j_{!}(\mathcal{M} \otimes "f^\lambda")$.*

(2) *For $\lambda \in \Lambda$ with $\lambda \ll 0$, the map $\tilde{\mathcal{M}}_\lambda \rightarrow j_*(\mathcal{M} \otimes "f^\lambda")$ is an isomorphism.*

(3) *For $\lambda \in \Lambda$ with $\lambda \gg 0$, the map $j_!(\mathcal{M} \otimes "f^\lambda") \rightarrow \tilde{\mathcal{M}}_\lambda$ is an isomorphism.*

Note that assertion of the theorem provides an algorithm for computing $j_!(\mathcal{M})$. Namely, we must pick any finitely generated submodule $\tilde{\mathcal{M}} \subset j_*(\mathcal{M} \otimes "f^s")$, such that $j^*(\tilde{\mathcal{M}}) \simeq \mathcal{M} \otimes "f^s"$, and

$$j_!(\mathcal{M}) \simeq \tilde{\mathcal{M}}/s - n$$

for a sufficiently large integer n .

2. A REFORMULATION

2.1. We shall derive Theorem 1 from a slightly more precise assertion. Before stating it, let us recall the following result, which is a well-known consequence of the b-function lemma (the proof will be recalled for completeness in the next section).

In what follows, if P is a module over $k[s]$ and λ is an element of $k \subset \text{Spec}(k[s])$, we shall denote by $P_{(\lambda)}$ the localization of P at the corresponding maximal ideal, i.e., $s - \lambda$.

We are going to study $D_X[s]_{(\lambda)}$ -submodules $\tilde{\mathcal{M}}_{(\lambda)} \subset j_*(\mathcal{M} \otimes "f^s")_{(\lambda)}$ such that $j^*(\tilde{\mathcal{M}}_{(\lambda)}) = (\mathcal{M} \otimes "f^s")_{(\lambda)}$. We shall denote this set by $\mathbf{V}(\mathcal{M}_{(\lambda)})$.

Theorem 2. *For any $\lambda \in k$ the following holds:*

(A) *The $D_X[s]_{(\lambda)}$ -module $j_*(\mathcal{M} \otimes "f^s")_{(\lambda)}$ is finitely generated. Denote it $\tilde{\mathcal{M}}_{(\lambda)}^{max}$.*

(B) *The set $\mathbf{V}(\mathcal{M}_{(\lambda)})$ contains the minimal element. Denote it $\tilde{\mathcal{M}}_{(\lambda)}^{min}$. Moreover, we have:*

(B.1) *The quotient $\tilde{\mathcal{M}}_{(\lambda)}^{max}/\tilde{\mathcal{M}}_{(\lambda)}^{min}$ is $(s - \lambda)$ -torsion.*

(B.2) *The natural map $j_!(\mathcal{M} \otimes "f^\lambda") \rightarrow (\tilde{\mathcal{M}}_{(\lambda)}^{min})/s - \lambda$ is an isomorphism.*

(C) *There exists a subset $\Lambda \subset k$ equal to the union of finitely many arithmetic progressions such for $\lambda \notin \Lambda$, $\tilde{\mathcal{M}}_{(\lambda)}^{min} = \tilde{\mathcal{M}}_{(\lambda)}^{max}$.*

2.2. The strengthening of Theorem 1 mentioned above reads as follows:

Theorem 3. *Let Λ be as above, and let $\tilde{\mathcal{M}}$ be an element of $\mathbf{V}(\mathcal{M})$.*

(I) *For $\lambda \notin \Lambda$, the maps*

$$\tilde{\mathcal{M}}_{(\lambda)}^{\min} \rightarrow \tilde{\mathcal{M}}_{(\lambda)} \rightarrow \tilde{\mathcal{M}}_{(\lambda)}^{\max}$$

are isomorphisms.

(II) *The map $\tilde{\mathcal{M}}_{(\lambda)} \rightarrow \tilde{\mathcal{M}}_{(\lambda)}^{\max}$ is an isomorphism for all $\lambda \in \Lambda$ that are $\ll 0$.*

(III) *The element $\tilde{\mathcal{M}}$ belongs to $\mathbf{V}_f(\mathcal{M})$ if and only if the map $\tilde{\mathcal{M}}_{(\lambda)}^{\min} \rightarrow \tilde{\mathcal{M}}_{(\lambda)}$ is an isomorphism for all $\lambda \in \Lambda$ that are $\gg 0$.*

2.3. Let us first see some obvious implications. First, point (C) of Theorem 2 implies point (I) of Theorem 3. Combined with point (B.2) of Theorem 2, point (I) of Theorem 3 implies point (1) of Theorem 1.

Point (II) of Theorem 3 implies point (2) of Theorem 1. Point (III) of Theorem 3, combined with point (B.2) of Theorem 2 implies point (3) of Theorem 1.

Finally, the "only if" direction Theorem 3(III), combined with point (A) of Theorem 2, implies the "if" direction.

Furthermore, we have the following corollaries:

Corollary 1. *Specifying an element $\tilde{\mathcal{M}} \in \mathbf{V}(\mathcal{M})$ is equivalent to specifying, for each $\lambda \in \Lambda$, of an element $\tilde{\mathcal{M}}_{(\lambda)} \in \mathbf{V}(\mathcal{M}_{(\lambda)})$, such that $\tilde{\mathcal{M}}_{(\lambda)} = \tilde{\mathcal{M}}_{(\lambda)}^{\max}$ for all λ that are $\ll 0$.*

Corollary 2. *Let $\tilde{\mathcal{M}}^1$ and $\tilde{\mathcal{M}}^2$ be elements of $\mathbf{V}_f(\mathcal{M})$. Then the localizations $\tilde{\mathcal{M}}_{(\lambda)}^1$ and $\tilde{\mathcal{M}}_{(\lambda)}^2$ coincide for all but finitely many elements $\lambda \in k$.*

2.4. We shall now give a description of the set $\mathbf{V}(\mathcal{M}_{(\lambda)})$, appearing in Corollary 1, in terms of a vanishing cycles datum. With no restriction of generality, we can assume that $\lambda = 0$.

Recall that Sect. 4.2 of [2] identifies the quotient $\tilde{\mathcal{M}}_{(0)}^{\max}/\tilde{\mathcal{M}}_{(0)}^{\min}$, which is a $D_X[s]_{(0)}$ -module set-theoretically supported on $Y = X - U$, with the D-module $\Psi^{\text{nilp}}(\mathcal{M})$ of nilpotent nearby cycles of \mathcal{M} , with the action of s on it being the nilpotent "logarithm of monodromy" operator.

Thus, elements \mathcal{N} of $\mathbf{V}(\mathcal{M}_{(0)})$ are in bijection with s -stable D_X -submodules

$$\mathcal{K} \subset \Psi^{\text{nilp}}(\mathcal{M}).$$

For each \mathcal{K} as above, let us describe more explicitly the corresponding D_X -module $\mathcal{N}_0 := \mathcal{N}/s$. By [1], \mathcal{N}_0 is completely determined by the corresponding D-module of vanishing cycles $\Phi^{\text{nilp}}(\mathcal{N}_0)$, together with maps

$$\Psi^{\text{nilp}}(\mathcal{M}) \xrightarrow{\mathbf{c}} \Phi^{\text{nilp}}(\mathcal{N}_0) \xrightarrow{\mathbf{v}} \Psi^{\text{nilp}}(\mathcal{M}),$$

such that the composition $\mathbf{v} \circ \mathbf{c} : \Psi^{\text{nilp}}(\mathcal{M}) \rightarrow \Psi^{\text{nilp}}(\mathcal{M})$ equals s .

It is easy to see that $\Phi^{\text{nilp}}(\mathcal{N}_0)$ is given in terms of \mathcal{K} by either of the following two expressions:

$$\text{coker} \left(\mathcal{K} \xrightarrow{\iota \oplus s} \Psi^{\text{nilp}}(\mathcal{M}) \oplus \mathcal{K} \right)$$

or

$$\ker \left(\Psi^{nilp}(\mathcal{M})/\mathcal{K} \oplus \Psi^{nilp}(\mathcal{M}) \xrightarrow{s \oplus \pi} \Psi^{nilp}(\mathcal{M})/\mathcal{K} \right),$$

where $\iota : \mathcal{K} \hookrightarrow \Psi^{nilp}(\mathcal{M})$ and $\pi : \Psi^{nilp}(\mathcal{M}) \rightarrow \Psi^{nilp}(\mathcal{M})/\mathcal{K}$ are the natural embedding and projection, respectively. The above kernel and co-kernel are identified by means of the map $\Psi^{nilp}(\mathcal{M}) \oplus \mathcal{K} \rightarrow \Psi^{nilp}(\mathcal{M})/\mathcal{K} \oplus \Psi^{nilp}$ which has the following non-zero components:

$$-s : \Psi^{nilp}(\mathcal{M}) \rightarrow \Psi^{nilp}(\mathcal{M}); \iota : \mathcal{K} \rightarrow \Psi^{nilp}(\mathcal{M}); \pi : \Psi^{nilp}(\mathcal{M}) \rightarrow \Psi^{nilp}(\mathcal{M})/\mathcal{K}.$$

The map \mathbf{c} is the composition

$$\Psi^{nilp}(\mathcal{M}) \rightarrow \Psi^{nilp}(\mathcal{M}) \oplus \mathcal{K} \rightarrow \Phi^{nilp}(\mathcal{N}_0),$$

and the map \mathbf{v} is the composition

$$\Phi^{nilp}(\mathcal{N}_0) \rightarrow \Psi^{nilp}(\mathcal{M})/\mathcal{K} \oplus \Psi^{nilp}(\mathcal{M}) \rightarrow \Psi^{nilp}(\mathcal{M}).$$

We note that the $!$ -restriction of \mathcal{N}_0 to Y is then

$$\mathrm{Cone}(\Psi^{nilp}(\mathcal{M})/\mathcal{K} \xrightarrow{s} \Psi^{nilp}(\mathcal{M})/\mathcal{K})[-1],$$

and the $*$ -restriction of \mathcal{N}_0 to Y is $\mathrm{Cone}(\mathcal{K} \xrightarrow{s} \mathcal{K})$.

3. PROOFS

3.1. As all statements are local, we can assume that X is affine. First, let us recall the statement of the usual b-function lemma:

Lemma 1. (J. Bernstein) *Let \mathcal{M} be as in Sect. 1.1, and let m_1, \dots, m_n be generators of \mathcal{M} as a D_U -module. Then there exist elements $P_{i,j} \in D_X[s]$ and an element $\mathbf{b} \in k[s]$ such that for every i*

$$\sum_j P_{i,j}(m_j \otimes f^s) = \mathbf{b} \cdot (m_i \otimes f^{s-1}).$$

Let us deduce some of the statements of Theorems 2 and 3:

3.2. First, it is clear that for $\lambda \in k$ and $n \in \mathbb{Z}$ such that

$$\left((\lambda - n) - \mathbb{N} \right) \cap \mathrm{roots}(\mathbf{b}) = \emptyset,$$

the elements $m_i \otimes f^{s-n}$ generate $j_*(\mathcal{M} \otimes "f^s")_{(\lambda)}$ as a $D_X[s]_{(\lambda)}$ -module. This implies point (A) of Theorem 2.

Set

$$\Lambda = \mathbb{Z} + \mathrm{roots}(\mathbf{b}).$$

Point (C) of Theorem 2 and point (II) of Theorem 3 follow as well.

3.3. Note that we also obtain that the $D_X \otimes k(s)$ -module $j_*(\mathcal{M} \otimes "f^s") \otimes_{k[s]} k(s)$ does not have proper submodules, whose restriction to U is $(\mathcal{M} \otimes "f^s") \otimes_{k[s]} k(s)$.

This proves point (B.1) of Theorem 2 modulo the existence of $\tilde{\mathcal{M}}_{(\lambda)}^{min}$.

3.4. To prove point (B) of Theorem 2 and the remaining "only if" direction of Theorem 3(III), we shall use a duality argument.

Let A be a localization of a smooth k -algebra (we shall take A to be either $k[s]$ or $k[s]_{(\lambda)}$, or $k(s)$). Let $n = \dim(X)$. Consider the ring $D_X \otimes A$.

Let $D_{coh}^b(D_X \otimes A\text{-mod})$ (resp., $D_{coh}^b(\text{mod-}D_X \otimes A)$) denote the bounded derived category of left (resp., right) $D_X \otimes A$ -modules with coherent cohomologies.

Consider the contravariant functor

$$\mathbb{D}_A : D_{coh}^b(D_X \otimes A\text{-mod}) \rightarrow D_{coh}^b(D_X \otimes A\text{-mod}),$$

defined by composing the contravariant functor

$$\mathcal{M} \mapsto \text{RHom}(\mathcal{M}, D_X \otimes A),$$

which maps

$$D_{coh}^b(D_X \otimes A\text{-mod}) \rightarrow D_{coh}^b(\text{mod-}D_X \otimes A),$$

followed by tensor product with $\omega_X^{-1}[n]$ that maps $D_{coh}^b(\text{mod-}D_X \otimes A)$ back to $D_{coh}^b(D_X \otimes A\text{-mod})$. The same argument as in the case of usual D-modules shows that $\mathbb{D}_A \circ \mathbb{D}_A \simeq \text{Id}$.

We have the following basic property of the functor \mathbb{D}_A : let $A \rightarrow B$ be a homomorphism between k -algebras, and let \mathcal{N} be an object of $D_{coh}^b(D_X \otimes A\text{-mod})$. We have:

$$(1) \quad \mathbb{D}_B \left(B \overset{L}{\otimes}_A \mathcal{N} \right) \simeq B \overset{L}{\otimes}_A \mathbb{D}_A(\mathcal{N}).$$

In particular, for $\mathcal{M} \in D_{coh}^b(D_X\text{-mod})$, we have $\mathbb{D}_A(\mathcal{M} \otimes A) \simeq \mathbb{D}(\mathcal{M}) \otimes A$, where \mathbb{D} denotes the usual duality on $D_{coh}^b(D_X\text{-mod})$.

3.5. First, let us note that $\mathbb{D}_{k[s]}(\mathcal{M} \otimes "fs")$ is acyclic off cohomological degree 0, and

$$\mathbb{D}_{k[s]}(\mathcal{M} \otimes "fs") \overset{\sigma}{\simeq} \mathbb{D}(\mathcal{M}) \otimes "fs",$$

where σ means that the action of $k[s]$ on the two sides differs by the automorphism $\sigma : k[s] \rightarrow k[s], \sigma(s) = -s$.

Let now \mathcal{N} be an element of $\mathbf{V}(\mathcal{M}_{(\lambda)})$; in particular, \mathcal{N} is finitely generated over $D_X[s]_{(\lambda)}$ by Theorem 2(A). We shall prove:

Lemma 2.

- (a) *The $D_X[s]_{(\lambda)}$ -module $\mathbb{D}_{k[s]_{(\lambda)}}(\mathcal{N})$ is concentrated in cohomological degree zero.*
- (b) *The canonical map*

$$\mathbb{D}_{k[s]_{(\lambda)}}(\mathcal{N}) \rightarrow j_* \left(\mathbb{D}_{k[s]_{(\lambda)}}((\mathcal{M} \otimes "fs")_{(\lambda)}) \right) \overset{\sigma}{\simeq} j_*(\mathbb{D}(\mathcal{M}) \otimes "fs")_{(-\lambda)}$$

is an injection.

For the proof of the lemma see Sect. 3.7 below.

3.6. End of proofs of the theorems. The above lemma implies point (B) of Theorem 2 and the "if" direction in Theorem 3(III):

For point (B) of Theorem 2, the sought-for submodule $\tilde{\mathcal{M}}_{(\lambda)}^{min}$ is given by

$$\mathbb{D}_{k[s]_{(\lambda)}}(j_*(\mathbb{D}(\mathcal{M}) \otimes "f^s")_{(-\lambda)}).$$

Point (B.2) follows from equation (1).

For a finitely generated submodule $\tilde{\mathcal{M}}$ as in point (III) of Theorem 3, the map

$$\tilde{\mathcal{M}}_{(\lambda)}^{min} \rightarrow \tilde{\mathcal{M}}_{(\lambda)}$$

is an isomorphism whenever the corresponding map

$$(\mathbb{D}_{k[s]}(\tilde{\mathcal{M}}))_{(-\lambda)} \rightarrow j_*(\mathbb{D}(\mathcal{M}) \otimes "f^s")_{(-\lambda)}$$

is an isomorphism.

3.7. Proof of Lemma 2. We shall use the following corollary of Lemma 1, established in [3]:

Corollary 3. *The $D_X \otimes k(s)$ -module $j_*(\mathcal{M} \otimes "f^s") \otimes_{k[s]} k(s)$ is holonomic.*

From the corollary, we obtain that non-zero cohomologies of $\mathbb{D}_{k[s]_{(\lambda)}}(\mathcal{N})$ are s -torsion. Hence, to prove point (a), it is enough to show that

$$(2) \quad k \bigotimes_{k[s]_{(\lambda)}}^L \mathbb{D}_{k[s]_{(\lambda)}}(\mathcal{N})$$

is acyclic off cohomological degree 0.

This acyclicity would also imply that $\mathbb{D}_{k[s]_{(\lambda)}}(\mathcal{N})$ has no s -torsion. Combined with Sect. 3.3, this would imply point (b) of the lemma as well.

Using isomorphism (1), the acyclicity of (2) is equivalent to $k \bigotimes_{k[s]_{(\lambda)}}^L \mathcal{N} =: \mathcal{N}_\lambda$ being holonomic. The latter is true for $\mathcal{N} = j_*(\mathcal{M} \otimes "f^s")_{(\lambda)}$, since in this case $\mathcal{N}_\lambda \simeq j_*(\mathcal{M} \otimes "f^\lambda")$, which is known to be holonomic.

For any \mathcal{N} we argue as follows. We note that $j_*(\mathcal{M} \otimes "f^s")_{(\lambda)}/\mathcal{N}$, being finitely generated over $D_X \otimes k[s]_{(\lambda)}$ and $(s - \lambda)$ -torsion, is finitely generated over D_X . Since $(j_*(\mathcal{M} \otimes "f^s")_{(\lambda)}/\mathcal{N})/(s - \lambda)$ is holonomic, being a quotient of $j_*(\mathcal{M} \otimes "f^s")_{(\lambda)}/(s - \lambda)$, we obtain that $j_*(\mathcal{M} \otimes "f^s")_{(\lambda)}/\mathcal{N}$ is itself holonomic as a D_X -module.

We have a map

$$\mathcal{N}_\lambda \rightarrow j_*(\mathcal{M} \otimes "f^\lambda"),$$

whose kernel and cokernel are subquotients of $j_*(\mathcal{M} \otimes "f^s")_{(\lambda)}/\mathcal{N}$, which implies that \mathcal{N}_λ is holonomic as well. \square

3.8. An alternative argument. We can prove that $\mathbb{D}_{k[s]_{(\lambda)}}(\mathcal{N})$ lies in cohomological degree 0 directly, without quoting Corollary 3. Namely, we have the following general assertion that follows from the usual Nakayama lemma:

Lemma 3. *Let B be a filtered k -algebra such that $\mathrm{gr}(B)$ is a commutative finitely generated algebra over k . Let R be a localization of a commutative finitely generated k -algebra at a maximal ideal \mathfrak{m} . Then if \mathcal{P} is a finitely generated $R \otimes B$ -module, such that $\mathcal{P}/\mathfrak{m} \cdot \mathcal{P} = 0$, then $\mathcal{P} = 0$.*

Hence, Lemma 3 implies that the acyclicity of (2) implies that $\mathbb{D}_{k[s]_{(\lambda)}}(\mathcal{N})$ lies in cohomological degree 0, i.e., point (a) of Lemma 2.

In particular, we can apply Lemma 2(a) to $j_*(\mathcal{M} \otimes "f^s")$, and isomorphism (1) to the homomorphism $k[s] \rightarrow k(s)$. We conclude that $\mathbb{D}_{k(s)} \left(j_*(\mathcal{M} \otimes "f^s") \otimes_{k[s]} k(s) \right)$ lies in cohomological degree 0, i.e., that $j_*(\mathcal{M} \otimes "f^s") \otimes_{k[s]} k(s)$ is holonomic. This reproves Corollary 3.

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